Lecture 4
The Electric Potential

Please set your Clicker to Channel 41
Session ID: PHY122S15
Electric Potential Energy

Because the electric force on a charge $q$ in the neighborhood of other point-like charges depends on the relative distances only ...
- (just like for gravity !!)

... we can assign a Potential Energy $U$ to the charge $q$, that will only depend on its spatial position

As for any potential energy, we are only concerned with changes in potential energy: $\Delta U_{elec}$ ...

$$\int \mathbf{F}_i \cdot d\mathbf{x} \simeq \overline{\mathbf{F}}_i \cdot \Delta \mathbf{x} \equiv W_i \equiv -\Delta U_i ; \quad \sum_i W_i = \Delta K$$

$$\Rightarrow W_{NC} = \sum_i \Delta U_i + \Delta K \equiv \Delta E_{mech}$$

From: $\mathbf{F}=ma$; see PHY121
Electric Work, Energy, & Potential

A charge \( q = 2.0 \, \mu\text{C} \) moves in a uniform electric field \( E = 3.0 \times 10^3 \, \text{N/C} \) (e.g. between the plates of a capacitor)

- calculate the change in (electric) potential energy \( \Delta U \) when the charge \( q \) is moved from point A to point B:

**Work:**

\[
W_{\text{elec}} = \int_{A}^{B} \mathbf{F}_{\text{elec}} \cdot d\mathbf{x} = \mathbf{F}_{\text{elec}} \cdot \mathbf{AB} = q \mathbf{E} \cdot \mathbf{AB} = q E d
\]

\[
W_{\text{elec}} = 2.0 \times 10^{-6} \times 3.0 \times 10^3 \times 5.0 \times 10^{-2} \, \text{Nm} = 3.0 \times 10^{-4} \, \text{J}
\]

- Note: as we anticipated for a conservative force, ....

- ... the Work only depends on the end points A and B!!

**Potential Energy** \( U \):

\[
W_{A \rightarrow B} = -\Delta U
\]

\[
\Delta U = -q E d = -3.0 \times 10^{-4} \, \text{J}
\]

**Potential** \( V \):

\[
\Delta V = \Delta U / q = -Ed = -150 \, \text{J/C} = -150 \, \text{V(olt)} \left( E = 3000 \, \text{V/m} \right)
\]
Moving “upstream” by 4.0 cm in a 2000 V/cm electric field, the voltage changes by … Volt?

Rank | Responses
--- | ---
1 | 8000
2 | -8000
3 | 500
4 | 80
5 | -500
6 | Other

Values: 8000, {7900,…
Value Matches: 221
Inventor of the electro-chemical battery

- A pile of cells made of a zinc electrode and a copper electrode, separated by cardboard saturated with sulfuric acid or a brine mixture of salt and water.

- The electrolyte exists in the form $2H^+$ and $SO_4^{2-}$. The zinc, which is higher than both copper and hydrogen in the electrochemical series, reacts with the negatively charged sulfate ($SO_4^{2-}$).

- The positively charged hydrogen ions (protons) capture electrons from the copper, forming bubbles of hydrogen gas, $H_2$. This makes the zinc rod the negative electrode and the copper rod the positive electrode.

- Essentially, the difference in electro-chemical ionic binding energies provides the charge-pumping action!

- We now have two terminals, and the current will flow if we connect them. The reactions in this cell are as follows:

  - **Zinc** $Zn \rightarrow Zn^{2+} + 2e^-$
  - **Sulfuric acid** $2H^+ + 2e^- \rightarrow H_2$
12 V Lead-Acid Battery

- In the charged state, each cell contains electrodes of elemental lead (Pb) and lead(IV) oxide (PbO₂) in an electrolyte of approximately 33.5% sulfuric acid (H₂SO₄).

- In the discharged state both electrodes turn into lead(II) sulfate (PbSO₄) and the electrolyte loses its dissolved sulfuric acid and becomes primarily water.

- Anode Reaction:
  \[ \text{Pb}(s) + \text{HSO}_4^- (aq) \rightarrow \text{PbSO}_4(s) + \text{H}^+(aq) + 2e^- \]

- Cathode Reaction:
  \[ \text{PbO}_2(s) + \text{HSO}_4^- (aq) + 3\text{H}^+(aq) + 2e^- \rightarrow \text{PbSO}_4(s) + 2\text{H}_2\text{O}(l) \]
What if $A'$ and $B$ are both in the “plane” perpendicular to the field $\mathbf{E}$ (the dotted black line)?

- In that case $A'B_x$ is zero and:

$$\Delta U = -qE A'B_x = 0 \text{ J} ; \quad \Delta V = \Delta U / q = 0 \text{ J/C} \equiv 0 \text{ V(olt)}$$

This is true for any pair of $A', B$ points on the same plane:
- the Potential (Energy $U/q$) is the same everywhere on this plane ... 

**Equipotential Plane**
- Equipotential surface is $\perp \mathbf{E}$ everywhere
- Field $\mathbf{E}$ is pointing along the “downward” gradient of the potential

All this works for any field configuration ...
A. For a point charge, the equipotential surfaces are cylinders ...
B. For a planar charge, the equipotential surfaces are cylinders ...
C. For a line charge, the equipotential surfaces are cylinders ...
D. None of the above ...

Select the correct answer ...
Recap:

Coulomb Force (point charges $q$, $Q$): \[ F = \frac{1}{4\pi \kappa \varepsilon_0} \frac{qQ}{r^2} \]

Field: \[ E \equiv \frac{F}{q} \]

examples:

\[ E_{\text{point or sphere}} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2}; \quad E_{\text{line or cylinder}} = \frac{1}{2\pi \varepsilon_0} \frac{Q}{r}; \quad E_{\text{single plane}} = \frac{1}{2\varepsilon_0} \frac{Q}{A}; \quad E_{\text{parallel plate capacitor}} = \frac{1}{\varepsilon_0} \frac{Q}{A} \]

Energy of a test charge $q$ in a uniform Electric Field $E$:

(e.g. between the plates of a charged capacitor)

\[ \Delta U_{\text{electric}} = -W_{\text{elec}} = -\int_{\text{path } D} \mathbf{F}_{\text{elec}} \cdot d\mathbf{s} = -F_{\text{elec}} D_{\parallel} = qED_{\parallel} \]

Potential difference $\Delta V$:

(“voltage difference” between terminals)

\[ |\Delta V_{\text{capacitor}}| \equiv \frac{|\Delta U_{\text{electric}}|}{q} \]
Equipotential Surfaces

(Equi)potential surfaces denote surfaces in space of equal potential.
- the surfaces are not necessarily planes! e.g. spheres for point charge

Equipotential lines indicate lines of constant potential that intersect a given surface
- e.g. lines of equal height on a topo-map
- Note: $V_G = U_G/m = gh$, with $g =$ constant!

Where the equipotential lines are close together, the field is large!
- Electric Field: $E = V/d$ (magnitude), with $d =$ separation between lines (along the slope) ...

- Cfr: Gravitational Field: $E_G = gh/d = g \sin \theta$, and $F_G = mE_G = mgsin\theta$!
Capacitance of a Capacitor

The **Capacitance** of a capacitor is a measure of the amount of charge $Q$ that a capacitor will hold for a given potential difference $\Delta V$ between the plates.

- **Capacitance $C$:**
- **Unit:** $C/V = F \text{ (arad)}$
- **For a Parallel-Plate capacitor:**

$$|\Delta V| = Ed = \frac{Q}{\varepsilon_0 A} d \quad \Rightarrow C_{PP} \equiv \frac{Q}{|\Delta V|} = Q \frac{\varepsilon_0 A}{Qd} = \varepsilon_0 \frac{A}{d}$$

Michael Faraday
(1791 – 1867)

Capacitors (pF – mF) and Capacitor Symbols
A 1.0 mF capacitor has a 10.0 V potential difference across its terminals; its charge is … mC

Rank Responses
1 10
2 0.1
3 10000
4 1E-05
5 0.01
6 Other

Values: 10, {9.9, 10... Value Matches: 278
Energy, Potential, Capacitance of a Point Charge

A charge \( q \) moves in the field of a point charge \( Q \)

- calculate the change in (electric) potential going from \( A \) to \( B \):

**Work:** \( W_{\text{elec}} \equiv \int_{A}^{B} \mathbf{F}_{\text{elec}} \cdot d\mathbf{s} = KqQ \int_{r_{A}}^{r_{B}} \frac{1}{r^2} \mathbf{r} \cdot d\mathbf{s} = KqQ \int_{r_{A}}^{r_{B}} \frac{1}{r^2} \, dr = K \frac{qQ}{r_{A}} - K \frac{qQ}{r_{B}} \)

- Note: as we anticipated for a conservative force, ....
- ... the Work only depends on the end points \( A \) and \( B \) !!

**Potential Energy \( U \):**

\[
W_{A\rightarrow B} = -\Delta U = -U_{B} + U_{A}
\]

thus: \( U_{A} = K \frac{qQ}{r_{A}} \implies U(r) = K \frac{qQ}{r} \)

**Potential \( V \):**

\[
V(r) \equiv U(r)/q = K \frac{Q}{r}
\]

\[ \implies \Delta V = KQ \left( \frac{1}{r_{B}} - \frac{1}{r_{A}} \right) \implies C \equiv \frac{Q}{\Delta V} = \ldots \]
Potential & Capacitance of a Conducting Sphere …

Because of the symmetry, the electric field of a conducting sphere with charge \( Q \) and that of a point charge \( Q \) are identical OUTSIDE of the sphere

- the sphere itself is, of course, an equipotential surface.

To charge a sphere, we must bring charge \( dQ' \) onto the sphere which already has some charge \( Q' \):

\[
U = \int_0^Q K \frac{Q'}{R} dQ' = \frac{1}{2} K \frac{Q^2}{R} = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2
\]

- self-energy!

Similarly for a charged capacitor:

\[
U = \int_0^Q \frac{Q'}{\varepsilon_0 A} dQ' = \frac{1}{2} \varepsilon_0 A \frac{Q^2}{R} = \frac{1}{2} Q \Delta V = \frac{1}{2} C \Delta V^2
\]

- the stored energy: \( U = \frac{1}{2} \varepsilon_0 \frac{A}{d} (Ed)^2 = \frac{1}{2} \varepsilon_0 E^2 \times \frac{A}{d} \) \( \times \) Volume !

- the energy density: \( u \equiv \frac{U}{\text{Volume}} = \frac{1}{2} \varepsilon_0 E^2 \) generally true!
Example Problem

A system of two uncharged metal spheres, spaced 20.0 cm (center to center) apart, has a capacitance of \( C = 26.0 \text{ pF} \).

- How much work would it take to move an amount of charge \( q = 16.0 \text{ nC} \) from one sphere to the other?

Solution:

- Clearly, a case of Work - Energy conversion ...
- the final energy \( U_f \) after the transfer of charge is:

\[
\Delta U = U_f - U_i = \frac{1}{2} CV^2 - 0 = \frac{q^2}{2C} = \frac{(16 \times 10^{-9} \text{ C})^2}{2 \times 26 \times 10^{-12} \text{ F}} = 4.92 \mu\text{J}
\]

- The net work expended equals the energy now stored in this capacitor: \( U = \frac{1}{2} CV^2 = 4.92 \mu\text{J} \)

- Note, the UNITS check out fine: \( \frac{C^2}{C/V} = VC = (\text{J/C})C = \text{J} \)
Example: Energy of a System of Charges

- Consider two charges $Q_1 = +5 \, \mu C$, $Q_2 = -8 \, \mu C$, 5 cm apart:
  Their energy is:
  $$U = U_{12} = Q_1 V_2 = Q_1 K \frac{Q_2}{r_{12}} = 5.0 \times 10^{-6} \times 9.0 \times 10^9 \frac{-8.0 \times 10^{-6}}{5.0 \times 10^{-2}} = -7.2 \, J$$

- Now, calculate the potential energy of a third charge $Q_3 = +6 \, \mu C$, 3 cm from $Q_1$, 4 cm from $Q_2$:
  $$U = U_{31} + U_{32} = Q_3 V_1 + Q_3 V_2 = Q_3 K \left( \frac{Q_1}{r_{31}} + \frac{Q_2}{r_{32}} \right)$$
  $$= 6.0 \times 10^{-6} \times 9.0 \times 10^9 \left( \frac{5.0 \times 10^{-6}}{3.0 \times 10^{-2}} + \frac{-8.0 \times 10^{-6}}{4.0 \times 10^{-2}} \right) = -1.80 \, J$$

Note: the potential energy of $Q_3$ infinitely far away is ZERO because $V(r) \propto 1/r$!

- The TOTAL energy of the SYSTEM of ALL 3 charges is:
  $$U = U_{12} + U_{31} + U_{32} = Q_1 V_2 + Q_3 V_1 + Q_3 V_2 = -7.2 - 1.8 = -9.0 \, J$$
Example: Fusion in the Sun

A proton has a diameter of approximately \( d = 1.6 \times 10^{-15} \text{ m} \). When protons in the Sun collide to this distance, fusion may happen:

\[ p + p \rightarrow D^+ + e^+ + v \quad (D^+ = \text{pn}) \]

What is temperature \( T \) of the Sun’s thermonuclear core?

- Energy of the two protons required to approach each other to distance \( d \):

\[ U = eV = e \left( \frac{Ke}{d} \right) = e \times (9.0 \times 10^5 \text{ J/C}) \]

- Thus, assuming the protons collided head-on with equal initial speeds:

\[ K_i = 2 \times \frac{1}{2} m_p v^2 = U_f = eV \quad \Rightarrow \quad v = \sqrt{\frac{eV}{m_p}} = 9.2 \times 10^6 \text{ m/s} \]

- The protons form a plasma (ionized gas), and its temperature \( T \) is a measure of the average kinetic energy \( K \) of the protons:

\[ eV = \bar{K} = \frac{1}{2} m_p \bar{v}^2 = \frac{3}{2} k_B T \quad \Rightarrow \quad T = \frac{m_p \bar{v}^2}{3k_B} \approx \frac{2eV}{3k_B} = 3.4 \times 10^9 \text{ K} \]
A Dielectric material is an insulator which contains polarizable molecules ...

- Typically, the polarization field $E_{\text{pol}}$ partially opposes the external field $E$.
- The magnitude of the polarization field is proportional to the external field ...

Hence, inserting a dielectric material between the plates of a capacitor decreases the NET electric field between the plates:

$$E' \equiv \frac{E}{\kappa}$$ with $\kappa$ the dielectric constant for the material.

- Decreases the potential $\Delta V = Ed$ by the same factor $\kappa$: $\Delta V' = \Delta V / \kappa$.
- And increases the capacitance by factor $\kappa$:

$$C' \equiv \frac{Q}{\Delta V'} = \kappa \epsilon_0 \frac{A}{d}$$

Dielectrics are taken into account by the substitution: $\epsilon_0 \rightarrow \kappa \epsilon_0$. 

\[02/05/2015\] Lecture 4
Summary Electro-Statics:

Coulomb Force (for point charges): \( F = \frac{1}{4\pi\kappa\varepsilon_0} \frac{qQ}{r^2} \)

Field: \( E \equiv \frac{F}{q} \)

examples:
- \( E_{\text{point or sphere}} = \frac{1}{4\pi\kappa\varepsilon_0} \frac{Q}{r^2} \);
- \( E_{\text{single plane}} = \frac{1}{2\kappa\varepsilon_0} \frac{Q}{A} \);
- \( E_{\text{capacitor}} = \frac{1}{\kappa\varepsilon_0} \frac{Q}{A} \)

Potential:
- \( \Delta V_{\text{point or sphere}} = \frac{1}{4\pi\kappa\varepsilon_0} \frac{Q}{r} \);
- \( \Delta V_{\text{capacitor}} = \frac{1}{\kappa\varepsilon_0} \frac{Qd}{A} \)

Capacitance:
- \( C \equiv \frac{Q}{\Delta V} \);

Dielectrics:
- \( \varepsilon_0 \to \kappa\varepsilon_0 \)

Energy:
- \( \Delta U \equiv q\Delta V \);
- \( U_{\text{charged sphere}} = \frac{1}{2} K \frac{Q^2}{R} = \frac{1}{2} CV^2 \);
- \( U_{\text{charged capacitor}} = \frac{1}{2} CV^2 \);

Energy density:
- \( u = \frac{1}{2} \kappa\varepsilon_0 E^2 \);