PHY121 – Physics for the Life Sciences I

Lecture 21

1. Oscillations
2. Simple Harmonic Motion

Note: set your Clicker to Channel 21
Periodic Motion

Periodic motion is motion that repeats itself over and over again ...
- it is closely related to circular motion, as we will see a bit later...
- in uniform circular motion the $x$ and $y$ coordinates oscillate ...

Periodic motion is characterized by:
- **Amplitude** $A$ of the motion (positive by definition)
- The repetition **Period** $T \ [s]$, or
- the repetition **Frequency** $f \ [\text{cycles/s = Hz (Hertz)} = s^{-1}] = 1/T$

Necessary pre-condition for periodic motion: the existence of a NET RESTORING FORCE
The period $T$ of a 60 Hz Alternating Current (AC) signal is ...

A. 60 s
B. 30 s
C. $1/60$ s
D. $2/60$ s

$T = 1/f = 1/(60 \text{ Hz}) = 1/60 \text{ s}$
Simple Harmonic Motion (SHM) results when the magnitude of the net restoring force is simply proportional to the displacement:

\[ F_{Net} = -kx = ma = m \frac{d^2x}{dt^2} \]

or:

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x \]

A mass \( m \) hanging on an ideal spring is a good example.

- To find the equation of motion \( x(t) \) we solve the 2nd order differential equation above...
- Trial and error (& experience) gives the trial solution with arbitrary constants \( A, \omega \) and \( \phi \):

\[ x(t) = A \cos \theta(t) = A \cos(\omega t + \phi) = A \cos(2\pi \frac{t}{T} + \phi) = A \cos(2\pi f t + \phi) \]

- where \( \omega \) follows from substituting this trial solution:

\[ \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi) = -\frac{k}{m} x = -\frac{k}{m} A \cos(\omega t + \phi) \Rightarrow \omega = \sqrt{\frac{k}{m}} \]

  - units: rad/s; check: \([k/m] = [N/m/kg=\text{s}^{-2}] \Rightarrow \text{angular velocity!} \)
  - with amplitude \( A \) and “phase angle” \( \phi \) [rad] from “initial conditions”...
Example

Mass \( m = 2.0 \text{ kg} \) hangs off an ideal spring \( (k = 288 \text{ N/m}) \)

Initial conditions: I move \( m \) down \( x_1=0.10 \text{ m} \) from the equilibrium position, and then release it...

Calculate the equation of motion:

- Take \( x \) positive down from the equilibrium position...
- Note that gravity does not matter here; it is permanently cancelled out by the stretch \( x_0 \) the spring has in equilibrium \( mg - kx_0 = 0 \); thus gravity can be ignored if we consider the motion with respect to the new equilibrium point \( x = x_0 \).

- Then: \( x(t) = A \cos(\omega t + \varphi) = (0.10 \text{ m}) \cos(12t + \varphi) \)

- where \( \omega = \sqrt{k/m} = 12 \text{ rad/s} \), and where \( \varphi = 0 \) because the mass starts from rest at maximum \( x = x_1 = A \) (like a cosine)

- I know the motion; e.g. I can calculate the velocity at \( t=1 \text{ s} \):

\[
v(t) \equiv \frac{dx}{dt} = \frac{d}{dt} [A \cos(\omega t + \varphi)] = -\omega A \sin(\omega t + \varphi)\bigg|_{t=1 \text{ s}} = -1.2 \sin(12) = 0.64 \text{ m/s}
\]

- Calculate when/where the acceleration is maximum

\[
a \equiv \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} [A \cos(\omega t + \varphi)] = -\omega^2 A \cos(\omega t + \varphi) = -\omega^2 x \quad \Rightarrow \quad a_{\text{max}} = \omega^2 A = 14.4 \text{ m/s}^2
\]

i.e. when \( \omega t = \pi \) \( \Rightarrow \ t = \frac{3.14}{12} = 0.26 \text{ s} \)
Initial Conditions

“Initial Conditions”, i.e. the initial position and velocity, determine the Amplitude $A$ and the phase angle $\phi$ of the SHM: $x(t) = A \cos(\omega t + \phi)$  $\Rightarrow$  $v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$

- The initial position: $x_0 = A \cos \phi$;  the initial velocity: $v_0 = -\omega A \sin \phi$

\[
\begin{align*}
x_0 &= A \cos \phi \quad \Leftrightarrow \quad \phi = \arctan \left( -\frac{v_0}{\omega x_0} \right) \\
v_0 &= A \sin \phi \quad \Leftrightarrow \quad A = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}
\end{align*}
\]

Example: mass $m = 0.20$ kg on spring $k = 20$ N/m is set into motion at $x_0 = 0.30$ m and $v_0 = -4.0$ m/s; calculate the motion:

- $\omega = \sqrt{k/m} = \sqrt{20 \text{ kg s}^{-2} / 0.20 \text{ kg}} = \sqrt{100 \text{ s}^{-2}} = 10$ rad/s
- $A^2 = 16/100 + 9/100 = 25/100 \Rightarrow A = 0.50$ m
- $\phi = \arctan\left( -(-4.0)/(10 \times 0.30) \right) = \arctan(4.0/3.0) = 53^\circ = 0.93$ rad.

- Result: $x = (0.50 \text{ m}) \cos(10t + 0.93)$
why do we chose $A \cos(\omega t + \phi)$ instead of $A \sin(\omega t + \phi')$ to describe a SHM?

A. No particular reason; they are equal to each other with $\phi = \phi' + 90^\circ$ ...

B. Because SHM always starts ($t=0$) away from the equilibrium position $x=0$ ...

C. The cosine function is more basic than the sine function ...

D. The cosine function is easier to use than the sine function ...
Connection between SHM and Uniform Circular Motion

Uniform Circular Motion: consider a rotating position vector \( \mathbf{A} \) (a “phasor”)

The “projection” of uniform circular motion is SHM!

- rotation angle \( \theta \) for constant \( \omega = \omega_0 \) (i.e. zero \( \alpha \)),
- and starting phase angle \( \phi = \theta(t=0) = \theta_0 \):

\[
\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 = \phi + \omega t
\]
Summary Variables:

Characteristic of the motion:
- frequency \( f \), period \( T \), or the angular frequency \( \omega \):
- all related: \( f = 1/T \), \( \omega = 2\pi f = 2\pi/T \)
- frequency \( f \), period \( T \), angular frequency \( \omega \): all depend on the physics details:
  - e.g. mass-on-a spring: \( \omega = \sqrt{(k/m)} \)

From “Initial Conditions”:
- Amplitude \( A \)
- Phase angle \( \phi \)

Simple Harmonic Motion: \( x = A \cos(\omega t + \phi) \)
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- Note (Fourier Theorem): Any repetitive motion can be expressed as a sum of sine and cosine functions!
  \[ f(\omega t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \]

- Many applications in math, engineering, electronics, acoustics and music, optics, image and signal processing, econometrics, ....
Energy in SHM

Energy in SHM: the *mass-plus-spring* system again:

- at any time \( t \), the mass \( m \) has a velocity \( v(t) \) and a position \( x(t) \) (w.r.t the equilibrium position)

- Choosing the equilibrium position as the convenient point where spring potential energy is zero, and assuming friction/drag to be negligible, we get:

- Total energy: \( E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} kx_{\text{max}}^2 = \frac{1}{2} kA^2 = \frac{1}{2} mv_{\text{max}}^2 \)

- Check:

\[
E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) + \frac{1}{2} m \left[ -\omega A \sin(\omega t + \phi) \right]^2
\]

  - with \( \omega^2 = k/m \), this becomes simply:

\[
E = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) + \frac{1}{2} m \frac{k}{m} A^2 \sin^2(\omega t + \phi) = \frac{1}{2} kA^2
\]

  - and, of course, \( A \) and \( \phi \) can be expressed in the parameters of the initial conditions \( x_0 \) and \( v_0 \)

Note, kinetic and potential energy are functions of \( t \),

- and are *continuously changing* over each period, while the sum stays constant (in case of zero damping)
To fully describe a system exhibiting simple harmonic motion (without friction), I always need (in addition to $\omega$, $f$, or $T$)…

A. the position and speed at $t=0$ …
B. the position and speed at some known time $t$ …
C. either two positions or two speeds at different times …
D. any of the above …

We have 2 unknowns ($A$, $\varphi$) and thus we need two equations; any two will do …
Simple Pendulum

The simple pendulum: a single point-like mass $m$ suspended on a massless string of length $L$ will do SHM (for small amplitudes):

- the only ingredients for determining the period are $m$, $L$ and gravitational acceleration $g$, nothing else can matter!
- The period $T$ has units of $[s]$, $m$ is in $[kg]$, $L$ in $[m]$, and $g$ in $[m/s^{-2}]$. Thus, dimensionally: $T \propto \sqrt{(L/g)}$
- Full calculation: ($T$≡Tension or Period!)

$$F_{rad} = T - Mg \cos \theta = Ma_{rad} = M \frac{\nu^2}{L}$$

$$F_{tan} = -Mg \sin \theta$$

$$\tau_{Net} = -LMg \sin \theta = I \alpha = ML^2 \alpha = ML^2 \frac{d^2 \theta}{dt^2}$$

- The last expression, for small angle $\theta$, becomes:

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta \approx -\frac{g}{L} \theta$$

- with solution: $\theta(t) = A \cos(\omega t + \phi)$,
- and $\omega = \sqrt{g/L}$, the “natural frequency” of the system ...
- Thus: $T = 2\pi/\omega = 2\pi \sqrt{(L/g)}$
At which of the times in the figure below is kinetic energy being transformed to potential energy?

1. A & E & I
2. B & F
3. C & G
4. D & H

Look for the situations where $K$ is diminishing and $U_G$ increasing ...
Example

Tomato hornworms turn into remarkable moths called hawkmoths whose flight resembles that of a hummingbird.

To a good approximation, the wings move with simple harmonic motion with a very high frequency—about 26 Hz, a high enough frequency to produce an audible tone. The tips of the wings move up and down by about 5.0 cm from their central position during one cycle.

Given these numbers,

A. What is the maximum velocity of the tip of a hawkmoth wing? \( \omega = 2\pi f = 163 \text{ rad/s} \); \( v \equiv dx/dt = -\omega A\sin(\omega t + \varphi) \)

\[
\nu_{\text{max}} = \omega A = 163 \text{ rad/s} \times 0.050 \text{ m} = 8.2 \text{ m/s}
\]

B. What is the maximum acceleration of the tip of a hawkmoth wing? \( a \equiv dv/dt = -\omega^2 A\cos(\omega t + \varphi) \)

\[
a_{\text{max}} = \omega^2 A = (163 \text{ rad/s})^2 \times 0.050 \text{ m} = 1.33 \times 10^3 \text{ m/s}^2
\]
The physical pendulum is a pendulum formed by an extended object suspended in a point \( L \) meter away from its center-of-gravity:

- Full calculation:

\[
\tau_{\text{Net}} = -LMg \sin \theta = I \alpha = I \frac{d^2 \theta}{dt^2}
\]

- for small angle \( \theta \), this becomes:

\[
\frac{d^2 \theta}{dt^2} = -\frac{LMg}{I} \sin \theta \approx -\left( \frac{LMg}{I} \right) \theta
\]

- with solution:

\[
\theta(t) = A \cos(\omega t + \varphi),
\]

- with:

\[
\omega = \sqrt{\frac{LMg}{I}}, \text{ the “natural frequency”}
\]

and thus:

\[
T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{LMg}}
\]

Note: for \( I = ML^2 \), we recover the simple pendulum …
On your first trip to Planet X you happen to take along a $m=200 \text{ g}$ mass, a $L=40 \text{ cm}$-long spring, a meter stick, and a stopwatch. You're curious about the acceleration due to gravity on Planet X, where ordinary tasks seem easier than on earth, but you can't find this information in your Visitor's Guide. One night you suspend the spring from the ceiling in your room and hang the mass from it. You find that the mass stretches the spring by $x_0=30.0 \text{ cm}$. You then pull the mass down $x=12.0 \text{ cm}$ and release it. With the stopwatch you find that 10.0 oscillations take 16.0 s.

- Can you satisfy your curiosity?

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{m}{mg/x_0}} = 2\pi \sqrt{\frac{x_0}{g}} \quad \Rightarrow \quad g = \frac{4\pi^2}{T^2} x_0$$