Lecture 22

1. Recap: Oscillations and Simple Harmonic Motion
2. Damped Oscillations
3. Forced Oscillations and Resonance

Note: set your Clicker to Channel 21
Periodic motion is motion that repeats itself over and over again ...
  - it is closely related to circular motion, as we will see a bit later...
  - in uniform circular motion the $x$ and $y$ coordinates oscillate ...

Periodic motion is characterized by:
  - **Amplitude** $A$ of the motion (positive by definition)
  - The repetition **Period** $T \, \text{[s]}$, or
  - the repetition **Frequency** $f \, \text{[cycles/s = Hz (Hertz) = s}^{-1}] = 1/T$
  - **Angular Frequency**:
    \[ \omega \equiv \frac{2\pi}{T} = 2\pi f \]

Necessary pre-condition for periodic motion: the existence of a NET RESTORING FORCE
A periodic motion has a period of 2 s. The angular frequency of this motion is ...

Rank | Responses
--- | ---
1 | 0.5
2 | 3.14
3 | 2
4 | 10
5 | 3
6 | Other

Values: 3.14, {3.14,... Value Matches: 43
Simple Harmonic Motion

Simple Harmonic Motion (SHM) results when the magnitude of the net restoring force is simply proportional to the displacement:

\[ F_{\text{Net}} = -kx = ma = m\frac{d^2x}{dt^2} \]

or:

\[ \frac{d^2x}{dt^2} = -\frac{k}{m}x \]

A mass \( m \) hanging on an ideal spring is a good example.

- To find the equation of motion \( x(t) \) we solve the 2\(^{nd}\) order differential equation above...

- **Trial and error** (& experience) gives the trial solution with arbitrary constants \( A, \omega \) and \( \varphi \):
  \[
  x(t) = A\cos(\theta(t)) = A\cos(\omega t + \varphi) = A\cos(2\pi \frac{t}{T} + \varphi)
  \]

- where \( \omega \) follows from substituting this trial solution:
  \[
  \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \varphi) = -\frac{k}{m}x = -\frac{k}{m}A\cos(\omega t + \varphi) \Rightarrow \omega = \sqrt{\frac{k}{m}}
  \]

  - units: rad/s; check: \([k/m] = [N/m/kg=s^{-2}] \Rightarrow \text{angular velocity!}

- with amplitude \( A \) and “phase angle” \( \varphi \) [rad] from “initial conditions”...
"Initial Conditions", i.e. the initial position and velocity, determine the Amplitude $A$ and the phase angle $\phi$ of the SHM: $x(t) = A \cos(\omega t + \phi)$
- The initial position: $x_0 = A \cos \phi$; the initial velocity: $v_0 = -\omega A \sin \phi$

\[
x_0 = A \cos \phi \\
v_0 = A \sin \phi
\]

\[
\phi = \tan \left( -\frac{v_0}{\omega x_0} \right)
\]

\[
A = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}
\]

Example: mass $m = 0.20$ kg on spring $k = 20$ N/m is set into motion at $x_0 = 0.30$ m and $v_0 = -4.0$ m/s; calculate the motion:
- $\omega = \sqrt{(k/m)} = \sqrt{(20 \text{ kg s}^{-2} / 0.20 \text{ kg})} = \sqrt{100 \text{ s}^{-2}} = 10 \text{ rad/s}$
- $A^2 = (4.0 \text{ ms}^{-1}/10 \text{ s}^{-1})^2 + (0.30 \text{ m})^2 = 0.25 \text{ m}^2 \Rightarrow A = 0.50 \text{ m}$
- $\phi = \tan(-(-4.0 \text{ ms}^{-1})/(10 \times 0.30 \text{ ms}^{-1})) = \tan(+4.0/3.0) = 53^\circ = 0.93 \text{ rad.}$
- Result: $x = (0.50 \text{ m}) \cos((10 \text{ s}^{-1})t + 0.93)$
A 2.0 kg mass hangs at rest on a long bungee cord with spring constant 8.0 N/m. Then, the mass is kicked so that it gets a upwards speed of 3.0 m/s. Calculate the amplitude of the resulting motion (m).

Available Responses:
1. 2
2. 1.5
3. 4
4. 3
5. 2.9
6. Other

Results:
- Rank 1: 2 (20%)
- Rank 2: 1.5 (15%)
- Rank 3: 4 (11%)
- Rank 4: 3 (7%)
- Rank 5: 2.9 (5%) 
- Rank 6: Other (43%)

Values: 1.5, {1.5, 1...}
Value Matches: 28
Summary Simple Harmonic Motion:

Characteristic of the motion:
- frequency $f$, period $T$, or the angular frequency $\omega$:
- all related: $f = 1/T$, $\omega = 2\pi f = 2\pi/T$
- frequency $f$, period $T$, (angular) frequency $\omega$: all depend on the physics details:
  - e.g. mass ($m$) on a spring ($k$): $\omega = \sqrt{k/m}$
  - e.g. pendulum (length $L$, mass $M$, and inertia $I$): $\omega = \sqrt{LMg/I}$

From "Initial Conditions":
- Amplitude $A$ $x_0 = A \cos \phi$
- Phase angle $\phi$ $\frac{v_0}{-\omega} = A \sin \phi$

\[
\begin{align*}
\phi &= \atan\left(-\frac{v_0}{\omega x_0}\right) \\
A &= \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}
\end{align*}
\]

Simple Harmonic Motion: $x = A \cos(\omega t + \phi)$
- this can look like a sine-wave with the proper choice of $\phi$ ...
- the Fourier Theorem tells us that ANY repetitive motion can be modeled as an (infinite) sum of cosine and sine functions of the base frequency and multiples of it ...
What system(s) will **not** exhibit Simple Harmonic Motion?

1. a mass hanging off a spring
2. a pencil balancing on its tip
3. a pendulum
4. a tall skyscraper
5. a see-saw

A pencil balancing on its tip, or a see-saw, do **not** oscillate when kicked ...
Tall skyscrapers oscillate in a strong wind ...
Energy in SHM

Energy in SHM: the mass-plus-spring system again:
- at any time $t$, the mass $m$ has a velocity $v(t)$ and a position $x(t)$ (w.r.t the equilibrium position).
- Choosing the equilibrium position as the convenient point where spring potential energy is zero, and assuming friction/drag to be negligible, we get:
  - Total energy: $E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} kx_{\text{max}}^2 = \frac{1}{2} kA^2 = \frac{1}{2} mv_{\text{max}}^2$
  - Check: $E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) + \frac{1}{2} m \left[ -\omega A \sin(\omega t + \phi) \right]^2$
    - with $\omega^2 = k/m$, this becomes simply: $E = \frac{1}{2} k A^2 \cos^2(\omega t + \phi) + \frac{1}{2} m \left[ -\omega A \sin(\omega t + \phi) \right]^2$
    - and, of course, $A$ and $\phi$ can be expressed in the parameters of the initial conditions $x_0$ and $v_0$

Note, kinetic and potential energy are functions of $t$,
- and are continuously changing over each period, while the sum stays constant (in case of zero damping)
A 2.0 kg mass on a long bungee cord with spring constant 8.0 N/m is seen oscillating up and down with an amplitude of 2.0 m. Calculate the total mechanical energy of the system (J)

\[ E = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2} kx_{\text{max}}^2 = \frac{1}{2} kA^2 = \frac{1}{2} mv_{\text{max}}^2 \]

Rank | Responses | Value Matches
--- | --- | ---
1 | 16 | 111
2 | 32 |
3 | 2 |
4 | 4 |
5 | 8 |
6 | Other |

Values: 16, \{16, 15\}

Value Matches: 111
Damped Oscillations

So far, we assumed friction and “damping” to be negligible, which was simplifying things enormously...

In the **special** case of small “damping” proportional to velocity \( v \), \( F_d \approx -bv \) (a good approximation for drag at small velocity), we can again solve the motion analytically:

\[
F_{\text{Net}} = -kx - bv = -kx - b \frac{dx}{dt} = ma = m \frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m} x - \frac{b}{m} \frac{dx}{dt}
\]

- this differential equation can again be solved, with solution:

\[
x(t) = Ae^{-\frac{b}{2m}t} \cos(\omega' \cdot t + \varphi), \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}
\]

- this is similar to SHM, now with a “damping factor” \( \exp(-b/2m) \) multiplying the amplitude \( A \),

- and a slightly different frequency \( \omega' \approx \omega_0 \) (\( \omega_0 \) is the natural frequency of the non-damped system)

- Amplitude and phase factor again depend on “initial conditions”...
Damped Oscillations

In the special case of small “damping” proportional to velocity $v$, $F_d \approx -bv$ (a good approximation for drag at small velocity), we can again solve the motion:

$$x(t) = \frac{Ae^{-\frac{bt}{2m}}}{x_{max}(t)} \cos(\omega' t + \phi)$$
with

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Note the units:
- units of $b$: N/(m/s) = Ns/m = kg/s
- units of $\tau$: kg/(kg/s) = s

A graph of $x_{max}$ as a function of time is an exponential decay.

$$x_{max}(t) = Ae^{-\frac{bt}{2m}} = Ae^{-t/\tau}, \quad \tau \equiv \frac{2m}{b}$$

The time constant $\tau$ is the time for the maximum displacement to decay to $1/e$ of its initial value.
Damped Oscillations

- "Critical Damping" occurs when: \( \omega' \equiv \sqrt{\frac{k}{m} - \left( \frac{b}{2m} \right)^2} = \sqrt{\omega_0^2 - \left( \frac{b}{2m} \right)^2} = 0 \)

  - i.e. for \( b = 2\sqrt{km} \)

- "Overdamping" occurs when \( b \) is even larger
  - and \( \omega' \) becomes imaginary ...

- The solution no longer oscillates, but is an exponential function:
  \[
  x(t) = C_1 e^{-c_1 t} + C_2 e^{c_2 t}
  \]

  - with \( C_1 \) and \( C_2 \) determined by initial conditions, and \( c_1 \) and \( c_2 \) functions of only \( k, m, \) and \( b \).

In damped oscillations, energy is no longer conserved, and decreases over time. The time rate of change of energy \( dE/dt \) (the "damping power") is quite simple to calculate:

\[
\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \right) = mv \frac{dv}{dt} + kx \frac{dx}{dt} = \nu (ma + kx)
\]

\[
F_{Net} = -kx - bv = ma \]

\[
= \nu (-bv) = -bv^2
\]
“Over-damping” occurs when …
(select the correct answer)

\[ x(t) = Ae^{\frac{-bt}{2m}} \cos(\omega' t + \varphi) , \text{ with } \omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \]

1. \( b > 2k \)
2. \( b > k \)
3. \( b > 2\sqrt{km} \)
4. \( b > 0 \)
5. \( b > 2m \)

Overdamping occurs when the number under the root-sign is negative; i.e. for \( b/(2m) > \sqrt{(k/m)} \) \( \Rightarrow b > 2\sqrt{(km)} \)
Forced Oscillations

Clearly, real-world oscillating systems need energy input if they are to continue oscillating.

- An example of an oscillatory system with an external force that does work on it, is a child on a swing being pushed by a parent.
- The parental driving force $F_d$ itself oscillates, ideally, in sync with the swing’s “natural frequency” $\omega^* \approx \omega_0 \equiv \sqrt{g/L}$ for best results...

Equation of Motion:

$$ F_{Net} = F_d(t) - kx - bv = F_d(t) - kx - b \frac{dx}{dt} = ma = m \frac{d^2x}{dt^2} $$

- thus: we must solve the (2nd order, inhomogeneous) differential equation:

$$ \frac{d^2x}{dt^2} = F_d(t) - \frac{k}{m} x - \frac{b}{m} \frac{dx}{dt} $$
Forced Oscillations

- For the special case $F_d = F_{\text{max}} \cos(\omega_d t)$ we can again solve this analytically!
- Now, the initial conditions do not matter (after a while), and all that matters is $F_{\text{max}}$ and $\omega_d$:

$$\frac{d^2 x}{dt^2} = \frac{F_d(t)}{m} - \frac{k}{m} x - \frac{b}{m} \frac{dx}{dt} \quad \Rightarrow \quad x = A \cos(\omega_d t + \varphi)$$

“trial solution”

- This motion has a pronounced “resonance” (for small damping $b$) for $\omega_d = \omega_0 \equiv \sqrt{(k/m)}$:

$$x = A \cos(\omega_d + \varphi), \quad \text{with} \quad A = \frac{F_{\text{max}}/m}{\sqrt{\left(\frac{k}{m} - \omega_d^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_d^2}}$$

Resonance is a very important and useful phenomenon (electronics, mechanics, etc.); sometimes a very destructive phenomenon: see the Collapse of the Tacoma Narrows Bridge on www.youtube.com
Resonance – derivation for those interested

Derivation of the resonance response curve of the motion when \( F_d = F_{\text{max}} \cos(\omega_d t) \):

\[
F_{\text{Net}} = F_d(t) - kx - bv = F_0 \cos \omega_d t - kx - b \frac{dx}{dt} = ma = m \frac{d^2x}{dt^2}
\]

\[
\frac{F_0}{m} \cos \omega_d t - \frac{k}{m} x - \frac{b}{m} \frac{dx}{dt} - \frac{d^2x}{dt^2} = 0
\]

- **Trial solution**: \( x = A \cos(\omega_d t + \varphi) \); fill in:

\[
\frac{F_0}{m} \cos \omega_d t - \frac{k}{m} A \cos(\omega_d t + \varphi) + \frac{b}{m} \omega_d A \sin(\omega_d t + \varphi) + \omega_d^2 A \cos(\omega_d t + \varphi) = 0
\]

- **Note**: \( \cos(\omega_d t + \varphi) = \cos \omega_d t \cos \varphi - \sin \omega_d t \sin \varphi \), \( \sin(\omega_d t + \varphi) = \sin \omega_d t \cos \varphi + \cos \omega_d t \sin \varphi \)

- **Collecting all terms with \( \cos \omega_d t \) and \( \sin \omega_d t \) separately**:

\[
0 = \cos \omega_d t \times \left[ \frac{F_0}{m} - \frac{k}{m} A \cos \varphi + \frac{b}{m} \omega_d A \sin \varphi + \omega_d^2 A \cos \varphi \right]
\]

\[
+ \sin \omega_d t \times \left[ \frac{k}{m} A \sin \varphi + \frac{b}{m} \omega_d A \cos \varphi - \omega_d^2 A \sin \varphi \right]
\]

- **because this is valid for all possible values of \( t \), this means that the bracketed expressions must be zero separately! This gives two equations with two unknowns \( \varphi \) and \( A \) (note: \( \omega_0^2 = k/m \)):

\[
\begin{align*}
(1) & \quad - \frac{F_0}{mA} = \left( \omega_d^2 - \omega_0^2 \right) \cos \varphi + \frac{b}{m} \omega_d \sin \varphi \\
(2) & \quad 0 = \left( \omega_d^2 - \omega_0^2 \right) \sin \varphi - \frac{b}{m} \omega_d \cos \varphi
\end{align*}
\]

\[
\begin{align*}
\tan \varphi &= \frac{b}{m} \omega_d \left/ \left( \omega_d^2 - \omega_0^2 \right) \right. \\
A &= \frac{F_0}{m} \sqrt{\left( \omega_d^2 - \omega_0^2 \right)^2 + \left( \frac{b}{m} \omega_d \right)^2}
\end{align*}
\]
Hearing is based partially on resonance: different regions in the inner ear respond by resonating with different frequencies of sound.

1. Sound waves enter the ear and cause the eardrum to vibrate.

2. Vibrations in the eardrum pass through a series of small bones...

3. ...to the cochlea, the sensing area of the inner ear...

4. ...where vibrations in the fluid drive vibrations in the basilar membrane.

To analyze the cochlea, we imagine the spiral structure unrolled, with the basilar membrane separating two fluid-filled chambers. The stapes, the last of the small bones, transfers vibrations into fluid in the cochlea.

As the distance from the stapes increases, the basilar membrane becomes wider and less stiff, so the resonance frequency of the membrane decreases.

\[ \omega_0 = 2\pi f = \sqrt{\frac{k}{m/A}} \]

stiffer \( \Rightarrow \) \( k \) larger

Oscillation amplitude of basilar membrane

Sounds of different frequencies cause different responses in the basilar membrane.
Clocks, Accuracy, and the Balance Wheel …

Temperature error

- A major remaining source of inaccuracy was the effect of temperature changes. An increase in temperature makes the spring and the balance get slightly longer from thermal expansion, but a more important effect is that the elasticity of the spring's metal decreases. The weaker spring takes longer to return the balance wheel back toward the center, so the 'beat' gets slower and the watch loses time. Ferdinand Berthoud found in 1773 that an ordinary brass balance and steel hairspring, subjected to a 60°F (33°C) temperature increase, loses 393 seconds (6.5 minutes) per day, of which 312 seconds is due to spring elasticity decrease. [15]

Temperature compensated balance wheels

- The need for an accurate clock for celestial navigation during sea voyages drove many advances in balance technology in 18th century Britain and France. Even a 1 second per day error in a marine chronometer could result in a 17-mile error in ship's position after a 2 month voyage. John Harrison was first to apply temperature compensation to a balance wheel in 1753, using a bimetallic 'compensation curb' on the spring, in the first successful marine chronometers. These achieved an accuracy of a fraction of a second per day. [16]

- A simpler solution was devised around 1765 by Pierre Le Roy, and improved by John Arnold and Thomas Earnshaw: the Earnshaw or compensating balance wheel. [16] The key was to make the balance wheel change size with temperature. If the balance's diameter could be made to shrink with temperature, the smaller moment of inertia would make it rotate faster, like a spinning ice skater that pulls in her arms. The faster balance would take less time to oscillate back and forth, compensating for the slowing caused by the weaker spring.

- To accomplish this, the outer rim of the balance was made of a 'sandwich' of two metals; a layer of steel on the inside fused to a layer of brass on the outside. Strips of this bimetallic construction bend toward the steel side when they are warmed, because the thermal expansion of brass is greater than steel. The rim was cut open at two points next to the spokes of the wheel, so it resembled an S-shape with two circular bimetallic 'arms'. Indeed, these wheels are sometimes referred to as "Z - balances". A temperature increase makes the arms bend inward toward the center of the wheel, and the shift of mass inward makes the balance spin faster, cancelling out the slowing due to the spring. The amount of compensation is adjusted by moveable weights on the arms. Marine chronometers with this type of balance had errors of only 3 - 4 seconds per day over a wide temperature range. [17] By the 1870s compensated balances began to be used in watches.
1940 Collapse of the Tacoma Narrows Bridge on www.youtube.com